

Approximate Kalman–Bucy filter for continuous-time semi-Markov jump linear systems

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Abstract

The aim of this paper is to propose a new numerical approximation of the Kalman–Bucy filter for semi-Markov jump linear systems. This approximation is based on the selection of typical trajectories of the driving semi-Markov chain of the process by using an optimal quantization technique. The main advantage of this approach is that it makes pre-computations possible. We derive a Lipschitz property for the solution of the Riccati equation and a general result on the convergence of perturbed solutions of semi-Markov switching Riccati equations when the perturbation comes from the driving semi-Markov chain. Based on these results, we prove the convergence of our approximation scheme in a general infinite countable state space framework and derive an error bound in terms of the quantization error and time discretization step. We employ the proposed filter in a magnetic levitation example with Markovian failures and compare its performance with both the Kalman–Bucy filter and the Markovian linear minimum mean squares estimator.

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1 Introduction

Markov jump linear systems (MJLS) have been largely studied and disseminated during the last decades. MJLS have a relatively simple structure that allows for useful, strong

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properties [9, 10, 14, 15], and provide suitable models for applications [13, 33, 32], with a booming field in web/internet based control [17, 20]. One limitation of MJLS is that the sojourn times between jumps is a time-homogeneous exponential random variable, thus motivating the study of a wider class of systems with general sojourn-time distributions, the so-called semi-Markov jump linear systems (sMJLS) or sojourn-time-dependent MJLS [20, 6, 31, 19, 21].

In this paper, we consider continuous-time sMJLS with instantaneous (or close to instantaneous) observation of the state of the semi-Markov chain at time instant t , denoted here by $\theta(t)$. The state space of the semi-Markov chain may be infinite. We seek for an approximate optimal filter for the variable $x(t)$ that composes the state of the sMJLS jointly with $\theta(t)$. Of course, estimating the state component $x(t)$ is highly relevant and allows the use of standard control strategies like linear state feedback.

It is well known that the optimal estimator for $x(t)$ is given by the standard Kalman–Bucy filter (KBF) [1, 22, 23, 24, 26] because, given the observation of the past values of θ , the distribution of the random variable $x(t)$ is exactly the same as in a time varying system. The main problem faced when implementing the KBF for MJLS or sMJLS, particularly in continuous time, is the pre-computation. Pre-computation refers to the computation of the relevant parameters of the KBF and storage in the controller/computer memory prior to the system operation, which makes the implementation of the filter fast enough to couple with a wide range of applications. Unfortunately, pre-computation is not viable for (s)MJLS in continuous time, as it involves solving a Riccati differential equation that branches at every jump time T_k , and the jumps can occur at any time instant according to an exponential distribution, so that pre-computation would involve computation of an infinite number of branches. Another way to explain this drawback of the KBF is to say that the KBF is not a Markovian linear estimator because the gain at time t does not depend only on $\theta(t)$ but on the whole trajectory $\{\theta(s), 0 \leq s \leq t\}$. This drawback of the KBF has motivated the development of other filters for MJLS, and one of the most successful ones is the Markovian linear minimum mean squares estimator (LMMSE) that has been derived in [16], whose parameters can be pre-computed, see also [10, 8]. To our best knowledge, there is no pre-computable filter for sMJLS.

The filter proposed here is built in several steps. The first step is the discretization by quantization of the Markov chain, providing a finite number of typical trajectories. The second step consists in solving the Riccati differential equation on each of these trajectories and store the results. To compute the filter in real time, one just needs to select the appropriate pre-computed branch at each jump time and follow it until the next jump time. This selection step is made by looking up the projection of the real jump time in the quantization grid and choosing the corresponding Riccati branch. In case the real jump time is observed with some delay (non-instantaneous observation of θ), then the observed jump time is projected in the quantization grid instead, see Remarks 4.7, 4.13.

The quantization technique selects optimized typical trajectories of the semi-Markov chain. Optimal quantization methods have been developed recently in numerical probability, nonlinear filtering or optimal stochastic control for diffusion processes with applications in finance [2, 3, 27, 28, 29, 30] or for piecewise deterministic Markov processes with applications in reliability [4, 5, 11, 12]. To our best knowledge, this technique has not been applied to MJLS or sMJLS yet. The optimal quantization of a random variable X consists in finding a finite grid such that the projection \hat{X} of X on this grid minimizes some L^p norm of the difference $X - \hat{X}$. Roughly speaking, such a grid will have more points in the areas of high density of X . One interesting feature of this procedure is that the construction of the optimized grids using the CLVQ algorithm (competitive learning vector

quantization) [27, 18] only requires a simulator of the process and no special knowledge about the distribution of X .

As explained for instance in [30], for the convergence of the quantized process towards the original process, some Lipschitz-continuity conditions are needed, hence we start investigating the Lipschitz continuity of solutions of Riccati equations. Of course, this involves evaluating the difference of two Riccati solutions, which is not a positive semi-definite nor a negative-definite matrix, preventing us to directly use the positive invariance property of Riccati equations, thus introducing some complication in the analysis given in Theorem 4.2. A by product of our procedure is a general result on the convergence of perturbed solutions of semi-Markov switching Riccati equations, when the perturbation comes from the driving semi-Markov chain and can be either a random perturbation of the jump times or a deterministic delay, or both, see Remark 4.7. Regarding the proposed filter, we obtain an error bound w.r.t. the exact KBF depending on the quantization error and time discretization step. It goes to zero when the number of points in the grids goes to infinity.

The approximation results are illustrated and compared with the exact KBF and the LMMSE in the Markovian framework for a numerical example of a magnetic suspension system, confirming via Monte Carlo simulation that the proposed filter is effective for state estimation even when a comparatively low number of points in the discretization grids is considered.

The paper is organized as follows. Section 2 presents the KBF and the sMJLS setup. The KBF approximation scheme is explained in Section 3, and its convergence is studied in Section 4. The results are illustrated in a magnetic suspension system, see Section 5, and some concluding remarks are presented in Section 6.

2 Problem setting

We start with some general notation. For $z, \hat{z} \in \mathbb{R}$, $z \wedge \hat{z} = \min\{z, \hat{z}\}$ is the minimum between z and \hat{z} . For a vector $X = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|X|$ denotes its Euclidean norm $|X|^2 = \sum x_i^2$ and X' denotes its transpose. Let $\mathcal{C}(n)$ be the set of $n \times n$ symmetric positive definite matrices and I_n (or I when there is no ambiguity) the identity matrix of size $n \times n$. For any two symmetric positive semi-definite matrices M and \widehat{M} , $M \geq \widehat{M}$ means that $M - \widehat{M}$ is positive semidefinite and $M > \widehat{M}$ means that $M - \widehat{M} \in \mathcal{C}(n)$. Let $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote the lowest and highest eigenvalue of matrix $M \in \mathcal{C}(n)$ respectively. For a matrix $M \in \mathbb{R}^{n \times n}$, M' is the transpose of M and $\|M\|$ stands for its L^2 matrix norm $\|M\|^2 = \lambda_{\max}(M'M)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, \mathbb{E} denote the expectation with respect to \mathbb{P} , and $Var(X)$ is the variance-covariance matrix of the random vector X . Let $\{\theta(t), t \geq 0\}$ be a semi-Markov jump process on the countable state space \mathcal{S} . We denote by F_i the cumulative distribution function of the sojourn time of θ in state i . For a family $\{M_i, i \in \mathcal{S}\}$ of square matrices indexed by \mathcal{S} , we set $\|M\|_{\mathcal{S}} = \sup_{i \in \mathcal{S}} \|M_i\| \leq \infty$.

We consider a sMJLS satisfying

$$\begin{cases} dx(t) &= A_{\theta(t)}x(t)dt + E_{\theta(t)}dw(t), \\ dy(t) &= C_{\theta(t)}x(t)dt + D_{\theta(t)}dv(t), \end{cases}$$

for $0 \leq t \leq T$, where T is a given time horizon, $(x(t), \theta(t)) \in \mathbb{R}^{n_1} \times \mathcal{S}$ is the state process, $y(t) \in \mathbb{R}^{n_2}$ is the measurement process, $\{w(t), 0 \leq t \leq T\}$ and $\{v(t), 0 \leq t \leq T\}$ are independent standard Wiener processes with respective dimensions n_3 and n_4 , independent from $\{\theta(t), t \geq 0\}$, and $\{A_i, i \in \mathcal{S}\}$, $\{C_i, i \in \mathcal{S}\}$, $\{D_i, i \in \mathcal{S}\}$ and $\{E_i, i \in \mathcal{S}\}$ are families of

matrices with respective size $n_1 \times n_1$, $n_2 \times n_1$, $n_2 \times n_4$ and $n_1 \times n_3$ such that $D_i D_i' > 0$ is nonsingular for all i (nonsingular measurement noise).

We use two different sets of assumptions for the parameters of our problems. The first one is more restrictive but relevant for applications, and the second more general one will be used in the convergence proofs.

Assumption 2.1 *The state space \mathcal{S} is finite, $\mathcal{S} = \{1, 2, \dots, N\}$ and the cumulative distribution functions of the sojourn times F_i are Lipschitz continuous with Lipschitz constant λ_i , $i \in \mathcal{S}$.*

Assumption 2.2 *The state space \mathcal{S} is countable, the quantities $\|A\|_{\mathcal{S}}$, $\|C\|_{\mathcal{S}}$, $\|D\|_{\mathcal{S}}$, $\|DD'\|_{\mathcal{S}}$ and $\|E\|_{\mathcal{S}}$ are finite. The cumulative distribution functions of the sojourn times F_i are Lipschitz continuous with Lipschitz constant λ_i , $i \in \mathcal{S}$ and*

$$\bar{\lambda} = \sup_{i \in \mathcal{S}} \{\lambda_i\} < \infty.$$

Note that the extra assumptions in the infinite case hold true automatically in the finite case, and that the Lipschitz assumptions hold true automatically for MJLS (i.e., when the distributions of F_i are exponential).

We address the filtering problem of estimating the value of $x(t)$ given the observations $\{y(s), \theta(s), 0 \leq s \leq t\}$ for $0 \leq t \leq T$. It is well-known that the KBF is the optimal estimator because the problem is equivalent to estimating the state of a linear time-varying system (with no jumps), taking into account that the past values of θ are available. The KBF satisfies the following equation

$$d\hat{x}_{KB}(t) = A_{\theta(t)} \hat{x}_{KB}(t) dt + K_{KB}(t) (dy(t) - C_{\theta(t)} \hat{x}_{KB}(t) dt),$$

for $0 \leq t \leq T$, with initial condition $\hat{x}_{KB}(0) = \mathbb{E}[x(0)]$ and gain matrix

$$K_{KB}(t) = P_{KB}(t) C_{\theta(t)}' (D_{\theta(t)} D_{\theta(t)}')^{-1}, \quad (1)$$

for $0 \leq t \leq T$, where $P_{KB}(t)$ is an $n_1 \times n_1$ matrix-valued process satisfying the Riccati matrix differential equation

$$\begin{cases} dP_{KB}(t) &= R(P_{KB}(t), \theta(t)) dt, \\ P_{KB}(0) &= \text{Var}(x(0)), \end{cases} \quad (2)$$

for $0 \leq t \leq T$, where $R : \mathbb{R}^{n_1 \times n_1} \times \mathcal{S} \rightarrow \mathbb{R}^{n_1 \times n_1}$ is defined for any $M \in \mathbb{R}^{n_1 \times n_1}$ and $i \in \mathcal{S}$ by

$$R(M, i) = A_i M + M A_i' + E_i E_i' - M C_i' (D_i D_i')^{-1} C_i M. \quad (3)$$

It is usually not possible to pre-compute a solution for this system (prior to the observation of $\theta(s)$, $0 \leq s \leq t$). Moreover, to solve it in real time after observing θ would require instantaneous computation of $P(t)$; one can obtain a delayed solution $P(t - \delta)$ where δ is the time required to solve the system, however using this solution as if it was the actual $P(t)$ in the filter may bring considerable error to the obtained estimate depending on δ and on the system parameters (e.g., many jumps may occur between $t - \delta$ and t).

The aim of this paper is to propose a new filter based on suitably chosen pre-computed solutions of Eq. (2) under the finiteness assumption 2.1 and to show convergence of our estimate to the optimal KBF when the number of discretization points goes to infinity under the more general countable assumption 2.2. We also compare its performance with the Fragosó-Costa LMMSE filter [16] on a real-world application.

3 Approximate Kalman–Bucy filter

The estimator is constructed as follows. We first select an optimized finite set of typical possible trajectories of $\{\theta(t), 0 \leq t \leq T\}$ by discretizing the semi-Markov chain and for each such trajectory we solve Eqs. (2), (1) and store the results. In real time, the estimate is obtained by looking up the pre-computed solutions and selecting the suitable gain given the current value of $\theta(t)$.

3.1 Discretization of the semi-Markov chain

The approach relies on the construction of optimized typical trajectories of the semi-Markov chain $\{\theta(t), 0 \leq t \leq T\}$. First we need to rewrite this semi-Markov chain in terms of its jump times and post-jump locations. Let $T_0 = 0$ and T_k be the k -th jump time of $\{\theta(t), 0 \leq t \leq T\}$ for $k \geq 1$,

$$T_k = \inf\{t \geq T_{k-1}; \theta(t) \neq \theta(T_{k-1})\}.$$

For $k \geq 0$ let $Z_k = \theta(T_k)$ be the post-jump locations of the chain. Let $S_0 = 0$ and for $k \geq 1$, $S_k = T_k - T_{k-1}$ be the inter-arrival times of the Markov process $\{\theta(t), 0 \leq t \leq T\}$. Using this notation, $\theta(t)$ can be rewritten as

$$\theta(t) = \sum_{k=0}^{\infty} Z_k \mathbb{1}_{\{T_k \leq t < T_{k+1}\}} = \sum_{k=0}^{\infty} Z_k \mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}}. \quad (4)$$

Under the finiteness assumption 2.1, as the state space \mathcal{S} of $\{\theta(t), 0 \leq t \leq T\}$ (and hence of $\{Z_k\}$) is finite, to obtain a fully discretized approximation of $\{\theta(t), 0 \leq t \leq T\}$ one only needs to discretize the inter-arrival times $\{S_k\}$ on a finite state space. One thus constructs a finite set of typical possible trajectories of $\{\theta(t), 0 \leq t \leq T\}$ up to a given jump time horizon T_n selected such that $T_n \geq T$ with high enough probability.

To discretize the inter-arrival times $\{S_k\}$, we choose a quantization approach that has been recently developed in numerical probability. Its main advantage is that the discretization is optimal in some way explained below. There exists an extensive literature on quantization methods for random variables and processes. The interested reader may for instance, consult the following works [2, 18, 27] and references therein. Consider X an \mathbb{R}^m -valued random variable such that $\mathbb{E}[|X|^2] < \infty$ and ν a fixed integer; the optimal L^2 -quantization of the random variable X consists in finding the best possible L^2 -approximation of X by a random vector \hat{X} taking at most ν different values, which can be carried out in two steps. First, find a finite weighted grid $\Gamma \subset \mathbb{R}^m$ with $\Gamma = \{\gamma^1, \dots, \gamma^\nu\}$. Second, set $\hat{X} = \hat{X}^\Gamma$ where $\hat{X}^\Gamma = \text{proj}_\Gamma(X)$ with proj_Γ denoting the closest neighbor projection on Γ . The asymptotic properties of the L^2 -quantization are given in e.g. [27].

Theorem 3.1 *If $\mathbb{E}[|X|^{2+\epsilon}] < +\infty$ for some $\epsilon > 0$ then one has*

$$\lim_{\nu \rightarrow \infty} \nu^{1/m} \min_{|\Gamma| \leq \nu} \mathbb{E}[|X - \hat{X}^\Gamma|^2]^{1/2} = C,$$

for some constant C depending only on m and the law of X and where $|\Gamma|$ denote the cardinality of Γ .

Therefore the L^2 norm of the difference between X and its quantized approximation \hat{X} goes to zero with rate $\nu^{-1/m}$ as the number of points ν in the quantization grid goes to infinity. The competitive learning vector quantization algorithm (CLVQ) provides the optimal grid based on a random simulator of the law of X and a stochastic gradient method.

In the following, we will denote by \hat{S}_k the quantized approximation of the random variable S_k and $\hat{T}_k = \hat{S}_1 + \dots + \hat{S}_k$ for all k .

3.2 Pre-computation of a family of solutions to Riccati equation

We start by rewriting the Riccati equation (2) in order to have a similar expression to Eq. (4). As operator R does not depend on time, the solution $\{P(t), 0 \leq t \leq T\}$ to Eq. (2) corresponding to a given trajectory $\{\theta(t), 0 \leq t \leq T\}$ can be rewritten as

$$P(t) = \sum_{k=0}^{\infty} P_k(t - T_k) \mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}},$$

for $0 \leq t \leq T$, where $\{P_0(t), 0 \leq t \leq T\}$ is the solution of the system

$$\begin{cases} dP_0(t) &= R(P_0(t), Z_0)dt, \\ P_0(0) &= p_0, \end{cases}$$

for $0 \leq t \leq T$, with $p_0 = \text{Var}(x(0))$, and for $k \geq 1$, $\{P_k(t), 0 \leq t \leq T\}$ is recursively defined as the solution of

$$\begin{cases} dP_k(t) &= R(P_k(t), Z_k)dt, \\ P_k(0) &= P_{k-1}(S_k). \end{cases}$$

Given the quantized approximation $\{\hat{S}_k\}$ of the sequence $\{S_k\}$, we propose the following approximations $\{\hat{P}_k(t), 0 \leq t \leq T\}$ of $\{P_k(t), 0 \leq t \leq T\}$ for all k . First, $\{\hat{P}_0(t), 0 \leq t \leq T\}$ is the solution of

$$\begin{cases} d\hat{P}_0(t) &= R(\hat{P}_0(t), Z_0)dt, \\ \hat{P}_0(0) &= p_0, \end{cases}$$

and for $k \geq 1$, $\{\hat{P}_k(t), 0 \leq t \leq T\}$ is recursively defined as the solution of

$$\begin{cases} d\hat{P}_k(t) &= R(\hat{P}_k(t), Z_k)dt, \\ \hat{P}_k(0) &= \hat{P}_{k-1}(\hat{S}_k). \end{cases}$$

Hence P_k and \hat{P}_k are defined with the same dynamics, the same horizon T , but different starting values, and all the \hat{P}_k can be computed off-line for each of the finitely many possible values of (Z_k, \hat{S}_k) (under the finiteness assumption 2.1 and for a finite number of jumps) and stored.

3.3 On line approximation

We suppose that on-line computations are made on a regular time grid with constant step δt . Note that in most applications δt is small compared to the time δ of instantaneous computation of $P(t)$. The state of the semi-Markov chain $\{\theta(t), 0 \leq t \leq T\}$ is observed, but the jumps can only be considered, in the filter operation, at the next point in the time grid. Set $\tilde{T}_0 = 0$, and for $k \geq 1$ define \tilde{T}_k as

$$\tilde{T}_k = \inf\{j; T_k < j\delta t\}\delta t,$$

hence \tilde{T}_k is the effective time at which the k -th jump is taken into account. One has $\tilde{T}_k > T_k$ and the difference between \tilde{T}_k and T_k is at most δt . We also set $\tilde{S}_k = \tilde{T}_k - \tilde{T}_{k-1}$ for $k \geq 1$. Now we construct our approximation $\{\tilde{P}(t), 0 \leq t \leq T\}$ of $\{P(t), 0 \leq t \leq T\}$ as follows

$$\tilde{P}(t) = \sum_{k=0}^{\infty} \hat{P}_k(t - \tilde{T}_k) \mathbb{1}_{\{0 \leq t - \tilde{T}_k < \tilde{S}_{k+1}\}} \mathbb{1}_{\{t \leq T\}}.$$

Thus we just select the appropriate pre-computed solutions and paste them at the approximate jumps times $\{\tilde{T}_k\}$, which can be done on-line. The approximate gain matrices are simply defined by

$$\tilde{K}(t) = \tilde{P}(t)C'_{\theta(t)}(D_{\theta(t)}D'_{\theta(t)})^{-1},$$

and the estimated trajectory satisfies

$$d\tilde{x}(t) = A_{\theta(t)}\tilde{x}(t)dt + \tilde{K}(t)(dy(t) - C_{\theta(t)}\tilde{x}(t)dt),$$

for $0 \leq t \leq T$, with initial condition $\tilde{x}(0) = \mathbb{E}[x(0)]$.

4 Convergence of the approximation procedure

The investigation of the convergence of our approximation scheme under the general assumption 2.2, is made in several steps again. The first one is the evaluation of the error between $P(t)$ and $\tilde{P}(t)$ up to the time horizon T and requires some Lipschitz regularity assumptions on the solution of Riccati equations. First, we establish these regularity properties. Then we derive the error between P and \tilde{P} , and finally we evaluate the error between the real KBF filter \hat{x}_{KB} and its quantized approximation \tilde{x} .

4.1 Regularity of the solutions of Riccati equations

For all $t \geq 0$, suitable matrix $p \in \mathcal{C}(n_1)$ and $i \in \mathcal{S}$ denote by $\phi_i(p, t)$ the solution at time t of the following Riccati equation starting from p at time 0,

$$\begin{cases} dP(t) &= R(P(t), i)dt, \\ P(0) &= p, \end{cases}$$

for $t \geq 0$. We start with a boundedness result.

Lemma 4.1 *Under Assumption 2.2, for all $\bar{p}_0 \in \mathcal{C}(n_1)$, there exist a matrix $\bar{p}_1 \in \mathcal{C}(n_1)$ such that $\bar{p}_1 \geq \bar{p}_0$ and for $p \leq \bar{p}_0$, $i \in \mathcal{S}$ and times $0 \leq t \leq T$, one has $\phi_i(p, t) \leq \bar{p}_1$.*

Proof. The Riccati equation can be rearranged in the following form

$$\begin{aligned} \frac{dP(t)}{dt} &= A_{aux}(t)P(t) + P(t)A_{aux}(t)' + E_iE_i' \\ &\quad + K_i(t)D_iD_i'K_i(t)', \end{aligned}$$

where $K_i(t) = P(t)C'_i(D_iD_i')^{-1}$ and $A_{aux}(t) = A_i - K_i(t)C_i$. For any matrix L with suitable dimensions, from the optimality of the KBF we have that $\phi_i(p, t) \leq \phi_L(p, t)$ where $\phi_L(p, t)$ is the covariance of a linear state observer with gain L , so that $\phi_L(p, t)$ is the solution of

$$\begin{aligned} \frac{dP(t)}{dt} &= (A_i - LC_i)(t)P(t) + P(t)(A_i - LC_i)' \\ &\quad + E_iE_i' + LD_iD_i'L', \\ P(0) &= p. \end{aligned}$$

In particular, we can set $L = 0$, and $\phi_L(p, t)$ is now the solution of the linear differential equation

$$\frac{dP(t)}{dt} = A_iP(t) + P(t)A_i' + E_iE_i', \quad P(0) = p, \quad (5)$$

which can be expressed in the form $\phi_L(p, t) = \Phi_1(t)p + \Phi_2(t)$ where $\Phi_1 \leq \beta e^{\alpha\|A_i\|t}\|p\|I$ and $\Phi_2 \leq \int_0^t \beta e^{\alpha\|A_i\|\tau}\|E_iE_i'\|I d\tau$ for some scalars α, β that do not depend on p, i . Set $\bar{p}_1 = \beta e^{\alpha T\|A\|S}(\|\bar{p}_0\|p_0 + T\|E\|_S^2 I)$, thus completing the proof. \square

Theorem 4.2 Under Assumption 2.2, for each $\tilde{p} \in \mathcal{C}(n_1)$ there exist $\ell, \eta > 0$ such that for all $i \in \mathcal{S}$ and $0 \leq t, \hat{t} \leq T$ and $p, \hat{p} \leq \tilde{p}$ one has

$$\|\phi_i(p, t) - \phi_i(\hat{p}, \hat{t})\| \leq \ell|t - \hat{t}| + \eta\|p - \hat{p}\|.$$

Proof. It follows directly from the definition of R in Eq. (3) that one has

$$\begin{aligned} & \frac{d\phi_i(p, t) - d\phi_i(\hat{p}, t)}{dt} \\ &= A_i\phi_i(p, t) + \phi_i(p, t)A_i' + E_iE_i' \\ & \quad - \phi_i(p, t)C_i'(D_iD_i')^{-1}C_i\phi_i(p, t) \\ & \quad - (A_i\phi_i(\hat{p}, t) + \phi_i(\hat{p}, t)A_i' + E_iE_i' \\ & \quad - \phi_i(\hat{p}, t)C_i'(D_iD_i')^{-1}C_i\phi_i(\hat{p}, t)) \\ &= A_i(\phi_i(p, t) - \phi_i(\hat{p}, t)) + (\phi_i(p, t) - \phi_i(\hat{p}, t))A_i' \\ & \quad - \phi_i(\hat{p}, t)C_i'(D_iD_i')^{-1}C_i(\phi_i(p, t) - \phi_i(\hat{p}, t)) \\ & \quad - (\phi_i(p, t) - \phi_i(\hat{p}, t))C_i'(D_iD_i')^{-1}C_i\phi_i(\hat{p}, t) \\ & \quad - (\phi_i(p, t) - \phi_i(\hat{p}, t))C_i'(D_iD_i')^{-1}C_i \\ & \quad \times (\phi_i(p, t) - \phi_i(\hat{p}, t)) \\ &= (A_i - \phi_i(\hat{p}, t)C_i'(D_iD_i')^{-1}C_i)(\phi_i(p, t) - \phi_i(\hat{p}, t)) \\ & \quad + (\phi_i(p, t) - \phi_i(\hat{p}, t))(A_i' - C_i'(D_iD_i')^{-1}C_i\phi_i(\hat{p}, t)) \\ & \quad - (\phi_i(p, t) - \phi_i(\hat{p}, t))C_i'(D_iD_i')^{-1}C_i \\ & \quad \times (\phi_i(p, t) - \phi_i(\hat{p}, t)), \end{aligned}$$

or, by denoting $X(t) = \phi_i(p, t) - \phi_i(\hat{p}, t)$, one has $X(0) = p - \hat{p}$ and

$$\begin{aligned} \frac{dX(t)}{dt} &= A_{aux}(t)X(t) + X(t)A_{aux}(t)' \\ & \quad - X(t)C_i'(D_iD_i')^{-1}C_iX(t), \end{aligned} \tag{6}$$

where we write $A_{aux}(t) = (A_i - \phi_i(\hat{p}, t)C_i'(D_iD_i')^{-1}C_i)$ for ease of notation. By setting $Y(0) = \|p - \hat{p}\|I \geq X(0)$ and using the order preserving property of the Riccati equation (6) it follows that $\{Y(t), 0 \leq t \leq T\}$ defined as the solution of

$$\begin{aligned} \frac{dY(t)}{dt} &= A_{aux}(t)Y(t) + Y(t)A_{aux}(t)' \\ & \quad - Y(t)C_i'(D_iD_i')^{-1}C_iY(t), \end{aligned} \tag{7}$$

satisfies $Y(t) \geq X(t)$ for all $t \geq 0$. The process $\{Y(t), 0 \leq t \leq T\}$ can be interpreted as the error covariance of a filtering problem¹, more precisely the covariance of the error $\hat{x}_{aux}(t) - x_{aux}(t)$ where $\{\hat{x}_{aux}(t), 0 \leq t \leq T\}$ satisfies

$$d\hat{x}_{aux} = A_{aux}(t)\hat{x}_{aux}dt + K(t)(dy - C_{aux}\hat{x}_{aux}dt),$$

with A_{aux} defined above, $C_{aux} = (C_i'(D_iD_i')^{-1}C_i)^{1/2}$, $\{K(t), 0 \leq t \leq T\}$ is the Kalman gain, and

$$\begin{cases} dx_{aux}(t) &= A_{aux}(t)x_{aux}(t)dt, \\ dy_{aux}(t) &= C_{aux}x_{aux}(t)dt + dv_{aux}(t), \end{cases}$$

¹Note that this does not hold true for the process $\{X(t), 0 \leq t \leq T\}$ as it may not be positive semidefinite.

where $\{v_{aux}(t), 0 \leq t \leq T\}$ is a standard Wiener process with incremental covariance $I dt$, and $x_{aux}(0)$ is a Gaussian random variable with covariance $p - \hat{p}$. Now, if we replace K with the (suboptimal) gain $L = 0$ we obtain a larger error covariance $Y_L(t) \geq Y(t)$. With the trivial gain $L = 0$ we also have

$$d\hat{x}_{aux} - dx_{aux} = A_{aux}(t)(\hat{x}_{aux} - x_{aux})dt,$$

so that direct calculation yields

$$\frac{dY_L(t)}{dt} = A_{aux}(t)Y_L(t) + Y_L(t)A_{aux}(t)', \quad (8)$$

with $Y_L(0) = \|p - \hat{p}\|I$. Recall that $\hat{p} \leq \tilde{p}$ by hypothesis, so that from Lemma 4.1 we get an uniform bound \bar{p}_1 for $\phi_i(\hat{p}, t)$, which in turn yields that $\|A_{aux}\|_{\mathcal{S}}$ is bounded in the time interval $0 \leq t \leq T$ and for all $\hat{p} \leq \tilde{p}$. This allows to write

$$Y(t) \leq \ell_1 \|p - \hat{p}\|I, \quad 0 \leq t \leq T,$$

for some $\ell_1 \geq 0$ (uniform on t, p, \hat{p} and i). Gathering some of the above inequalities together, one gets

$$\phi_i(p, t) - \phi_i(\hat{p}, t) = X(t) \leq Y(t) \leq Y_L(t) \leq \ell_1 \|p - \hat{p}\|I, \quad (9)$$

$0 \leq t \leq T$. Similarly as above, one can obtain

$$\phi_i(\hat{p}, t) - \phi_i(p, t) \leq \ell_2 \|p - \hat{p}\|I, \quad 0 \leq t \leq T, \quad (10)$$

where, again, ℓ_2 is uniform on t, p, \hat{p} and i . Eqs. (9), (10) and the fact that $\phi_i(\hat{p}, t) - \phi_i(p, t)$ is symmetric lead to

$$\begin{aligned} -\max(\ell_1, \ell_2) &\leq \lambda_{\min}(\phi_i(\hat{p}, t) - \phi_i(p, t)), \\ \lambda_{\min}(\phi_i(\hat{p}, t) - \phi_i(p, t)) &\leq \lambda_{\max}(\phi_i(\hat{p}, t) - \phi_i(p, t)) \\ &\leq \max(\ell_1, \ell_2). \end{aligned}$$

Hence, one has

$$\|\phi_i(\hat{p}, t) - \phi_i(p, t)\| \leq \max(\ell_1, \ell_2) \|p - \hat{p}\|,$$

completing the first part of the proof.

For the second part, similarly to the proof of the preceding lemma, we have that $\phi_i(p, t)$ is bounded from above by $X(t)$ the solution of the linear differential equation Eq.(5) with initial condition $X(0) = p$, and it is then simple to find scalars $\eta_1, \eta_2 > 0$ irrespective of i such that, for the entire time interval $0 \leq t \leq T$,

$$\|X(t) - p\|_{\mathcal{S}} \leq \|\Phi_1(t)\|_{\mathcal{S}} + \|(\Phi_2(t) - I)p\|_{\mathcal{S}} \leq \eta_1 t + \eta_2 t \|p\|.$$

Hence, one has

$$\phi_i(p, t) - p \leq X(t) - p \leq \|X(t) - p\|_{\mathcal{S}} I \leq (\eta_1 t + \eta_2 t \|p\|)I, \quad (11)$$

for all $t \geq 0$, leading to

$$\|\phi_i(p, t) - p\| \leq \eta_1 t + \eta_2 t \|p\|.$$

As $p \leq \tilde{p}$ by hypothesis, we have $\|p\| \leq n_1 \|\tilde{p}\|$ and it follows immediately from the above inequality that

$$\|\phi_i(p, t) - p\| \leq (\eta_1 + \eta_2 n_1 \|\tilde{p}\|)t. \quad (12)$$

As operator R does not depend on time, we have $\phi(p, t_1 + t_2) = \phi(\phi(p, t_1), t_2)$, $t_1, t_2 \geq 0$, and defining $\bar{p} = \phi(p, t_1)$, one has

$$\|\phi_i(p, t_1 + t_2) - \phi_i(p, t_1)\| = \|\phi_i(\bar{p}, t_2) - \bar{p}\|$$

and Eq. (12) allows to write

$$\|\phi_i(p, t_1 + t_2) - \phi_i(p, t_1)\| \leq (\eta_1 + \eta_2 n_1 \|\tilde{p}\|) t_2.$$

The result then follows by setting $t_1 = \hat{t}$ and $t_2 = t - \hat{t}$ if $t > \hat{t}$ or with $t_1 = t$ and $t_2 = \hat{t} - t$ otherwise. \square

4.2 Error derivation for gain matrices

We proceed in three steps. The first one is to study the error between $P_k(t)$ and $\hat{P}_k(t)$, the second step is to study the error between $P(t)$ and $\tilde{P}(t)$ and the last step is to compare the gain matrices $K_{KB}(t)$ and $\tilde{K}(t)$, for $0 \leq t \leq T$. We start with a preliminary important result that will enable us to use Theorem 4.2 in all the sequel.

Lemma 4.3 *Under Assumption 2.2, there exist a matrix $\bar{p} \in \mathcal{C}(n_1)$ such that for all integers $0 \leq k \leq n$ and times $0 \leq t \leq T$, one has*

$$P_k(t) \leq \bar{p}, \quad \hat{P}_k(t) \leq \bar{p}.$$

Proof. We prove the result by induction on k . For $k = 0$, one has $p_0 \in \mathcal{C}(n_1)$ and $P_0(t) = \hat{P}_0(t) = \phi_{Z_0}(p_0, t)$ for all $t \leq T$. Lemma 4.1 thus yields the existence of a matrix $\bar{p}_0 \in \mathcal{C}(n_1)$ such that $P_0(t) \leq \bar{p}_0$ for all $t \leq T$. Suppose that for a given $k \leq n-1$, there exists a matrix $\bar{p}_k \in \mathcal{C}(n_1)$ such that $P_k(t) \leq \bar{p}_k$ and $\hat{P}_k(t) \leq \bar{p}_k$ for all $t \leq T$. Then in particular, if $S_k \leq T$ and $\hat{S}_k \leq T$, one has $P_{k+1}(0) = P_k(S_k) \leq \bar{p}_k$ and $\hat{P}_{k+1}(0) = \hat{P}_k(\hat{S}_k) \leq \bar{p}_k$. Hence, Lemma 4.1 gives the existence of a matrix $\bar{p}_{k+1} \in \mathcal{C}(n_1)$ such that $P_{k+1}(t) \leq \bar{p}_{k+1}$ and $\hat{P}_{k+1}(t) \leq \bar{p}_{k+1}$ for all $t \leq T$. One thus obtains an increasing sequence (p_k) of matrices in $\mathcal{C}(n_1)$ and the result is obtained by setting $\bar{p} = \bar{p}_n$. \square

In the following, for \bar{p} given by Lemma 4.3 we set $\tilde{p} = \bar{p}$ in Theorem 4.2 and denote by $\bar{\ell}$ and $\bar{\eta}$ the corresponding Lipschitz constants. We now turn to the investigation of the error between the processes $P_k(t)$ and $\hat{P}_k(t)$.

Lemma 4.4 *Under Assumption 2.2, for all integers $0 \leq k \leq n$ and times $0 \leq t \leq T$, one has*

$$\|P_k(t) - \hat{P}_k(t)\| \leq \bar{\ell} \|P_{k-1}(S_k) - \hat{P}_{k-1}(\hat{S}_k)\|.$$

Proof. One has $P_k(t) = \phi_{Z_k}(P_{k-1}(S_k), t)$ and $\hat{P}_k(t) = \phi_{Z_k}(\hat{P}_{k-1}(\hat{S}_k), t)$. Hence, Lemma 4.3 and Theorem 4.2 yield

$$\begin{aligned} \|P_k(t) - \hat{P}_k(t)\| &= \|\phi_{Z_k}(P_{k-1}(S_k), t) - \phi_{Z_k}(\hat{P}_{k-1}(\hat{S}_k), t)\| \\ &\leq \bar{\ell} \|P_{k-1}(S_k) - \hat{P}_{k-1}(\hat{S}_k)\|, \end{aligned}$$

if $S_k, \hat{S}_k \leq T$, hence the result. \square

Lemma 4.5 *Under Assumption 2.2, for all integers $0 \leq k \leq n$ satisfying $S_k, \hat{S}_k \leq T$, one has*

$$\|P_k(S_{k+1}) - \hat{P}_k(\hat{S}_{k+1})\| \leq \sum_{j=0}^k \bar{\ell}^{k-j} \bar{\eta} |S_{j+1} - \hat{S}_{j+1}|.$$

Proof. By definition, one has $P_k(S_{k+1}) = \phi_{Z_k}(P_{k-1}(S_k), S_{k+1})$ and $\hat{P}_k(t) = \phi_{Z_k}(\hat{P}_{k-1}(\hat{S}_k), \hat{S}_{k+1})$. Hence as above, one has

$$\begin{aligned} & \|P_k(S_{k+1}) - \hat{P}_k(\hat{S}_{k+1})\| \\ &= \|\phi_{Z_k}(P_{k-1}(S_k), S_{k+1}) - \phi_{Z_k}(\hat{P}_{k-1}(\hat{S}_k), \hat{S}_{k+1})\| \\ &\leq \bar{\ell} \|P_{k-1}(S_k) - \hat{P}_{k-1}(\hat{S}_k)\| + \bar{\eta} |S_{k+1} - \hat{S}_{k+1}|. \end{aligned}$$

Then notice that one also has

$$\begin{aligned} & \|P_0(S_1) - \hat{P}_0(\hat{S}_1)\| \\ &= \|\phi_{Z_0}(p_0, S_1) - \phi_{Z_0}(p_0, \hat{S}_1)\| \leq \bar{\eta} |S_1 - \hat{S}_1|, \end{aligned}$$

and the result is obtained by recursion. \square

We can now turn to the error between the processes $P(t)$ and $\tilde{P}(t)$.

Theorem 4.6 *Under Assumption 2.2, for all $0 \leq t < T \wedge T_{n+1}$, one has*

$$\begin{aligned} & \mathbb{E}[\|P(t) - \tilde{P}(t)\|^2 \mathbb{1}_{\{0 \leq t \leq T \wedge T_{n+1}\}}]^{1/2} \\ & \leq \sum_{j=0}^{n-1} \bar{\ell}^{n-j} \bar{\eta} \mathbb{E}[|S_{j+1} - \hat{S}_{j+1}|^2]^{1/2} \\ & \quad + \bar{\eta} \delta t + n \|\bar{p}\| (\bar{\lambda} \delta t)^{1/2}, \end{aligned}$$

where \bar{p} is defined in Lemma 4.3.

Remark 4.7 *Note that the above result is very general. Indeed, we do not use in its proof that \hat{S}_k is the quantized approximation of S_k . We have established that, given a semi-Markov chain $\{\theta(t), 0 \leq t \leq T\}$ and a process $\{\hat{\theta}(t), 0 \leq t \leq T\}$ obtained by a perturbation of the jump times of $\{\theta(t), 0 \leq t \leq T\}$, the two solutions of the Riccati equations driven by these two processes respectively are not far away from each other, as long as the real and perturbed jump times are not far away from each other. We allow two kinds of perturbations, a random one, given by the replacement of S_k by \hat{S}_k and a deterministic one given by δt corresponding to a delay in the jumps. In the case of non-instantaneous observation of $\theta(t)$ (i.e., imperfect observation \tilde{S}_k of S_k), the difference $\mathbb{E}[|\tilde{S}_{j+1} - \hat{S}_{j+1}|^2]$ may not converge to zero but is still a valid upper bound for the approximation error of the Riccati solution and can reasonably be supposed small enough. Note also that the result is still valid for any L^q norm instead of the L^2 norm as the initial value of the Riccati solution is deterministic, as long as the distributions F_i have moments of order greater than q .*

Proof. By definition, one has for all $0 \leq t < T \wedge T_{n+1}$

$$\begin{aligned}
P(t) - \tilde{P}(t) &= \sum_{k=0}^n P_k(t - T_k) \mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}} \\
&\quad - \hat{P}_k(t - \tilde{T}_k) \mathbb{1}_{\{0 \leq t - \tilde{T}_k < \tilde{S}_{k+1}\}} \\
&= \sum_{k=0}^n (P_k(t - T_k) - \hat{P}_k(t - T_k)) \mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}} \\
&\quad + \sum_{k=0}^n (\hat{P}_k(t - T_k) - \hat{P}_k(t - \tilde{T}_k)) \mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}} \\
&\quad + \sum_{k=0}^n \hat{P}_k(t - \tilde{T}_k) (\mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}} - \mathbb{1}_{\{0 \leq t - \tilde{T}_k < \tilde{S}_{k+1}\}}) \\
&= \epsilon_1(t) + \epsilon_2(t) + \epsilon_3(t).
\end{aligned}$$

From Lemmas 4.4 and 4.5, the first term ϵ_1 can be bounded by

$$\begin{aligned}
\|\epsilon_1(t)\| &\leq \left\| \sum_{k=0}^n (P_k(t - T_k) - \hat{P}_k(t - T_k)) \mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}} \right\| \\
&\leq \sum_{k=0}^n \|P_k(t - T_k) - \hat{P}_k(t - T_k)\| \mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}} \\
&\leq \sum_{k=0}^n \ell \|P_{k-1}(S_k) - \hat{P}_{k-1}(\hat{S}_k)\| \mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}} \\
&\leq \sum_{k=0}^n \sum_{j=0}^{k-1} \bar{\ell}^{k-j} \bar{\eta} |S_{j+1} - \hat{S}_{j+1}| \mathbb{1}_{\{T_k \leq t < T_{k+1}\}} \\
&\leq \sum_{j=0}^{n-1} \bar{\ell}^{n-j} \bar{\eta} |S_{j+1} - \hat{S}_{j+1}|.
\end{aligned}$$

The second term ϵ_2 is bounded by Lemma 4.3 and Theorem 4.2 as follows

$$\begin{aligned}
\|\epsilon_2(t)\| &\leq \left\| \sum_{k=0}^n (\hat{P}_k(t - T_k) - \hat{P}_k(t - \tilde{T}_k)) \mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}} \right\| \\
&\leq \sum_{k=0}^n \|\hat{P}_k(t - T_k) - \hat{P}_k(t - \tilde{T}_k)\| \mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}} \\
&\leq \sum_{k=0}^n \bar{\eta} |T_k - \tilde{T}_k| \mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}} \\
&\leq \bar{\eta} \delta t,
\end{aligned}$$

using the fact that the difference between T_k and \tilde{T}_k is less than δt by construction. Finally, the last term ϵ_3 is bounded by using Lemma 4.3 and the fact that $0 \leq T_k \leq \tilde{T}_k$ for all k .

Indeed, one has

$$\begin{aligned}
& \mathbb{E}[\|\epsilon_3(t)\|^2]^{1/2} \\
& \leq \mathbb{E}\left[\left\|\sum_{k=0}^n \hat{P}_k(t - \tilde{T}_k)(\mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}} - \mathbb{1}_{\{0 \leq t - \tilde{T}_k < \tilde{S}_{k+1}\}})\right\|^2\right]^{1/2} \\
& \leq \|\bar{p}\| \sum_{k=0}^n \mathbb{E}[\|\mathbb{1}_{\{0 \leq t - T_k < S_{k+1}\}} - \mathbb{1}_{\{0 \leq t - \tilde{T}_k < \tilde{S}_{k+1}\}}\|^2]^{1/2} \\
& \leq \|\bar{p}\| \sum_{k=0}^n \mathbb{P}(t - \delta t \leq T_k \leq t)^{1/2} \\
& \leq n\|\bar{p}\| \sum_{i \in \mathcal{S}} (\lambda_i \delta t)^{1/2} \mathbb{P}(Z_k = i) \\
& \leq n\|\bar{p}\| (\bar{\lambda} \delta t)^{1/2}.
\end{aligned}$$

One obtains the result by taking the L^2 expectation norm also on both sides of the inequalities involving ϵ_1 and ϵ_2 . \square

Therefore, as the errors $\mathbb{E}[|S_{j+1} - \hat{S}_{j+1}|^2]$ go to 0 as the number of points in the discretization grids goes to infinity, we have the convergence of $\tilde{P}(t)$ to $P(t)$ as long as the time grid step δt also goes to 0. Theorem 4.6 also gives a convergence rate for $\|P(t) - \tilde{P}(t)\|$, providing that $0 \leq t < T \wedge T_{n+1}$. The convergence rate for the gain matrices is now straightforward from their definitions.

Corollary 4.8 *Under Assumption 2.2, for all $0 \leq t < T \wedge T_{n+1}$, one has*

$$\begin{aligned}
& \mathbb{E}[\|K_{KB}(t) - \tilde{K}(t)\|^2 \mathbb{1}_{\{0 \leq t \leq T \wedge T_{n+1}\}}]^{1/2} \\
& \leq \|C'(DD')^{-1}\|_{\mathcal{S}} \left(\sum_{j=0}^{n-1} \bar{\ell}^{n-j} \bar{\eta} \mathbb{E}[|S_{j+1} - \hat{S}_{j+1}|^2]^{1/2} \right. \\
& \quad \left. + \bar{\eta} \delta t + n\|\bar{p}\| (\bar{\lambda} \delta t)^{1/2} \right).
\end{aligned}$$

4.3 Error derivation for the filtered trajectories

We now turn to the estimation of the error between the exact KBF trajectory and our approximate one. We start with introducing some new notation. Let $b : \mathbb{R} \times \mathbb{R}^{2n_1} \rightarrow \mathbb{R}^{2n_1}$ and $\tilde{b} : \mathbb{R} \times \mathbb{R}^{2n_1} \rightarrow \mathbb{R}^{2n_1}$ be defined by

$$\begin{aligned}
b(t, z) &= \begin{pmatrix} A_{\theta(t)} & 0 \\ K_{KB}(t)C_{\theta(t)} & A_{\theta(t)} - K_{KB}(t)C_{\theta(t)} \end{pmatrix} z, \\
\tilde{b}(t, z) &= \begin{pmatrix} A_{\theta(t)} & 0 \\ \tilde{K}(t)C_{\theta(t)} & A_{\theta(t)} - \tilde{K}(t)C_{\theta(t)} \end{pmatrix} z
\end{aligned}$$

Let also $\sigma : \mathbb{R} \rightarrow \mathbb{R}^{2n_1 \times (n_3 + n_4)}$ and $\tilde{\sigma} : \mathbb{R} \rightarrow \mathbb{R}^{2n_1 \times (n_3 + n_4)}$ be defined by

$$\begin{aligned}
\sigma(t) &= \begin{pmatrix} E_{\theta(t)} & 0 \\ 0 & K_{KB}(t)D_{\theta(t)} \end{pmatrix}, \\
\tilde{\sigma}(t) &= \begin{pmatrix} E_{\theta(t)} & 0 \\ 0 & \tilde{K}(t)D_{\theta(t)} \end{pmatrix}.
\end{aligned}$$

Finally, set $W(t) = (w(t)', v(t)')'$, $X(t) = (x(t)', \hat{x}_{KB}(t)')'$ and $\tilde{X}(t) = (x(t)', \tilde{x}(t)')'$, so that the two processes $\{X(t), 0 \leq t \leq T\}$ and $\{\tilde{X}(t), 0 \leq t \leq T\}$ have the following dynamics

$$\begin{cases} dX(t) = b(t, X_t)dt + \sigma(t)dW(t), \\ X(0) = (x(0)', \mathbb{E}[x(0)']')', \\ \\ d\tilde{X}(t) = \tilde{b}(t, \tilde{X}_t)dt + \tilde{\sigma}(t)dW(t), \\ \tilde{X}(0) = (x(0)', \mathbb{E}[x(0)']')'. \end{cases}$$

The regularity properties of functions b , \tilde{b} , σ and $\tilde{\sigma}$ are quite straightforward from their definition.

Lemma 4.9 *Under Assumption 2.2, for all $0 \leq t \leq T$ and $z, \hat{z} \in \mathbb{R}^{2n_1}$, one has*

$$\begin{aligned} |b(t, z)| &\leq (\|A\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}}^2 \|(DD')^{-1}\|_{\mathcal{S}}) |z|, \\ |\tilde{b}(t, z)| &\leq (\|A\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}}^2 \|(DD')^{-1}\|_{\mathcal{S}}) |z|, \\ \|\sigma(t)\|_2 &\leq \|E\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}} \|(DD')^{-1}\|_2 \|D\|_{\mathcal{S}}, \\ \|\tilde{\sigma}(t)\|_2 &\leq \|E\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}} \|(DD')^{-1}\|_{\mathcal{S}} \|D\|_{\mathcal{S}}, \\ |b(t, z) - b(t, \hat{z})| &\leq (\|A\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}}^2 \|(DD')^{-1}\|_{\mathcal{S}}) |z - \hat{z}|, \\ |\tilde{b}(t, z) - \tilde{b}(t, \hat{z})| &\leq (\|A\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}}^2 \|(DD')^{-1}\|_{\mathcal{S}}) |z - \hat{z}|, \end{aligned}$$

where \bar{p} is the matrix defined in Lemma 4.3.

Proof. Upper bounds for $\|K_{KB}(t)\|_2$ and $\|\tilde{K}(t)\|_2$ come from the upper bounds for $P_k(t)$ and $\hat{P}_k(t)$ given in Lemma 4.3. \square

In particular, the processes $\{X(t), 0 \leq t \leq T\}$ and $\{\tilde{X}(t), 0 \leq t \leq T\}$ are well defined and $\mathbb{E}[\sup_{t \leq T} |X(t)|^2]$ and $\mathbb{E}[\sup_{t \leq T} |\tilde{X}(t)|^2]$ are finite, see e.g. [25]. Set also $\Delta(t) = K_{KB}(t) - \tilde{K}(t)$. In order to compare $X(t)$ and $\tilde{X}(t)$, one needs first to be able to compare b with \tilde{b} and σ with $\tilde{\sigma}$. The following result is straightforward from their definition.

Lemma 4.10 *Under Assumption 2.2, for all $0 \leq t \leq T$ and $z \in \mathbb{R}^{2n_1}$, one has*

$$\begin{aligned} |b(t, z) - \tilde{b}(t, z)| &\leq 2\|C\|_{\mathcal{S}} \|\Delta(t)\| |z|, \\ \|\sigma(t) - \tilde{\sigma}(t)\|_{\mathcal{S}} &\leq \|D\|_{\mathcal{S}} \|\Delta(t)\|. \end{aligned}$$

We also need some bounds on the conditional moments of $\{X(t), 0 \leq t \leq T\}$. Let $\{\mathcal{F}_t, 0 \leq t \leq T\}$ be the filtration generated by the semi-Markov process $\{\theta(t), 0 \leq t \leq T\}$, and $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_t]$.

Lemma 4.11 *Under Assumption 2.2, there exists a constant c_2 independent of the parameters of the system such that for $0 \leq t \leq T$ one has*

$$\begin{aligned} \mathbb{E}_T[\sup_{t \leq T \wedge T_{n+1}} |X(t)|^2] &\leq 2c_2 T (\|E\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}} \|(DD')^{-1}\|_{\mathcal{S}} \|D\|_{\mathcal{S}})^2 \\ &\quad \times \exp(2T^2 (\|A\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}}^2 \|(DD')^{-1}\|_{\mathcal{S}})^2). \end{aligned}$$

Proof. As $\{\theta(t), 0 \leq t \leq T\}$ and the noise sequence $\{W(t), 0 \leq t \leq T\}$ are independent, and the process $\{K_{KB}(t), 0 \leq t \leq T\}$ is only dependent on $\{\theta(t), 0 \leq t \leq T\}$ by construction,

one has

$$\begin{aligned}
& \mathbb{E}_T \left[\sup_{u \leq t \wedge T \wedge T_{n+1}} |X(u)|^2 \right] \\
& \leq 2\mathbb{E}_T \left[\sup_{u \leq t \wedge T \wedge T_{n+1}} \left| \int_0^u \sigma(s) dW(s) \right|^2 \right] \\
& \quad + 2\mathbb{E}_T \left[\sup_{u \leq t \wedge T \wedge T_{n+1}} \left| \int_0^u b(s, X(s)) ds \right|^2 \right] \\
& \leq 2c_2 \mathbb{E}_T \left[\int_0^{T \wedge T_{n+1}} \|\sigma(s)\|^2 ds \right] \\
& \quad + 2T \mathbb{E}_T \left[\int_0^{t \wedge T \wedge T_{n+1}} |b(s, X(s))|^2 ds \right],
\end{aligned}$$

from convexity and Burkholder–Davis–Gundy inequalities, see e.g. [25], where c_2 is a constant independent of the parameters of the problem. From Lemma 4.9 one gets

$$\begin{aligned}
& \mathbb{E}_T \left[\sup_{u \leq t \wedge T \wedge T_{n+1}} |X(u)|^2 \right] \\
& \leq 2c_2 T (\|E\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}} \|(DD')^{-1}\|_{\mathcal{S}} \|D\|_{\mathcal{S}})^2 \\
& \quad + 2T (\|A\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}}^2 \|(DD')^{-1}\|_{\mathcal{S}})^2 \\
& \quad \times \int_0^t \mathbb{E}_T \left[\sup_{u \leq s \wedge T \wedge T_{n+1}} |X(u)|^2 \right] ds.
\end{aligned}$$

Finally, we use Gronwall's lemma to obtain

$$\begin{aligned}
& \mathbb{E}_T \left[\sup_{t \leq T \wedge T_{n+1}} |X(t)|^2 \right] \\
& \leq 2c_2 T (\|E\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}} \|(DD')^{-1}\|_{\mathcal{S}} \|D\|_{\mathcal{S}})^2 \\
& \quad \times \exp(2T^2 (\|A\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}}^2 \|(DD')^{-1}\|_{\mathcal{S}})^2)
\end{aligned}$$

which proves the result. □

In the sequel, let \bar{X} be the upper bound given by Lemma 4.11:

$$\begin{aligned}
\bar{X} &= 2c_2 T (\|E\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}} \|(DD')^{-1}\|_{\mathcal{S}} \|D\|_{\mathcal{S}})^2 \\
& \quad \times \exp(2T^2 (\|A\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}}^2 \|(DD')^{-1}\|_{\mathcal{S}})^2).
\end{aligned}$$

We can now state and prove our convergence result.

Theorem 4.12 *Under Assumption 2.2, for $0 \leq t \leq T$ one has*

$$\mathbb{E}[|X(t) - \tilde{X}(t)|^2 \mathbb{1}_{\{0 \leq t \leq T \wedge T_{n+1}\}}] \leq \bar{c}_1 \exp(T\bar{c}_2),$$

with

$$\begin{aligned}
\bar{c}_1 &= (2\|D\|_{\mathcal{S}} + 8T\|C\|_{\mathcal{S}}^2 \bar{X}) \|C'_i(D_i D'_i)^{-1}\|_{\mathcal{S}} \\
& \quad \times \left(\sum_{j=0}^{n-1} \bar{\ell}^{n-j} \bar{\eta} \mathbb{E}[|S_{j+1} - \hat{S}_{j+1}|^2]^{1/2} \right. \\
& \quad \left. + \bar{\eta} \delta t + n \|\bar{p}\| (\bar{\lambda} \delta t)^{1/2} \right)^2, \\
\bar{c}_2 &= 2T (\|A\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}}^2 \|(DD')^{-1}\|_{\mathcal{S}})^2.
\end{aligned}$$

Proof. We follow the same lines as in the previous proof. As $\{\theta(t), 0 \leq t \leq T\}$ and the noise sequence $\{W(t), 0 \leq t \leq T\}$ are independent, and the processes $\{K_{KB}(t), 0 \leq t \leq T\}$ and $\{\tilde{K}(t), 0 \leq t \leq T\}$ are only dependent on $\{\theta(t), 0 \leq t \leq T\}$ by construction, one has

$$\begin{aligned}
& \mathbb{E}_T[|X(t) - \tilde{X}(t)|^2 \mathbb{1}_{\{0 \leq t \leq T \wedge T_{n+1}\}}] \\
& \leq 2\mathbb{E}_T\left[\left|\int_0^{t \wedge T \wedge T_{n+1}} (\sigma(s) - \tilde{\sigma}(s)) dW(s)\right|^2\right] \\
& \quad + 2\mathbb{E}_T\left[\left|\int_0^{t \wedge T \wedge T_{n+1}} (b(s, X(s)) - \tilde{b}(s, \tilde{X}(s))) ds\right|^2\right] \\
& \leq 2\mathbb{E}_T\left[\int_0^{t \wedge T \wedge T_{n+1}} \|\sigma(s) - \tilde{\sigma}(s)\|^2 ds\right] \\
& \quad + 2T\mathbb{E}_T\left[\int_0^{t \wedge T \wedge T_{n+1}} |b(s, X(s)) - \tilde{b}(s, \tilde{X}(s))|^2 ds\right],
\end{aligned}$$

from the isometry property of Itô integrals and Cauchy–Schwartz inequality. From Lemmas 4.9, 4.10 and Fubini one gets

$$\begin{aligned}
& \mathbb{E}_T[|X(t) - \tilde{X}(t)|^2 \mathbb{1}_{\{0 \leq t \leq T \wedge T_{n+1}\}}] \\
& \leq 2\|D\|_{\mathcal{S}} \int_0^{t \wedge T \wedge T_{n+1}} \|\Delta(s)\|^2 ds \\
& \quad + 2T\|C\|_{\mathcal{S}}^2 \int_0^{t \wedge T \wedge T_{n+1}} \|\Delta(s)\|^2 |\mathbb{E}_T[|X(s)|^2] ds \\
& \quad + 2T(\|A\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}}^2 \|(DD')^{-1}\|_{\mathcal{S}})^2 \\
& \quad \times \mathbb{E}_T\left[\int_0^{t \wedge T \wedge T_{n+1}} |X(s) - \tilde{X}(s)|^2 ds\right] \\
& \leq (2\|D\|_{\mathcal{S}} + 8T\|C\|_{\mathcal{S}}^2 \bar{X}) \int_0^{t \wedge T \wedge T_{n+1}} \|\Delta(s)\|^2 ds \\
& \quad + 2T(\|A\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}}^2 \|(DD')^{-1}\|_{\mathcal{S}})^2 \\
& \quad \times \int_0^t \mathbb{E}_T[|X(s) - \tilde{X}(s)|^2 \mathbb{1}_{\{0 \leq s \leq T \wedge T_{n+1}\}}] ds \\
& \leq \tilde{c}_1 + \tilde{c}_2 \int_0^t \mathbb{E}_T[|X(s) - \tilde{X}(s)|^2 \mathbb{1}_{\{0 \leq s \leq T \wedge T_{n+1}\}}] ds,
\end{aligned}$$

from Lemma 4.11, with

$$\begin{aligned}
\tilde{c}_1 &= (2\|D\|_{\mathcal{S}} + 8T\|C\|_{\mathcal{S}}^2 \bar{X}) \int_0^{t \wedge T \wedge T_{n+1}} \|\Delta(s)\|^2 ds, \\
\tilde{c}_2 &= 2T(\|A\|_{\mathcal{S}} + \|\bar{p}\| \|C\|_{\mathcal{S}}^2 \|(DD')^{-1}\|_{\mathcal{S}})^2.
\end{aligned}$$

We use Gronwall's lemma to obtain

$$\mathbb{E}_T[|X(t) - \tilde{X}(t)|^2 \mathbb{1}_{\{0 \leq t \leq T \wedge T_{n+1}\}}] \leq \tilde{c}_1 \exp(T\tilde{c}_2),$$

and conclude by taking the expectation on both sides and using Corollary 4.8 to bound $\mathbb{E}[\tilde{c}_1]$. \square

As a consequence of the previous result, $|\hat{x}_{KB}(t) - \tilde{x}(t)|$ goes to 0 almost surely as the number of points in the discretization grids goes to infinity.

Remark 4.13 *As noted in Remark 4.7, in the case of imperfect observation \tilde{S}_k of S_k , the errors $\mathbb{E}[|\tilde{S}_{j+1} - \hat{S}_{j+1}|^2]$ do not necessarily go to 0 if θ is not instantaneously observed, however the errors are small when the time delays are small. The previous result implies that the filter performance deterioration is proportional to these errors. Acceptable performances can still be achieved in applications where θ is not instantaneously observed.*

5 Numerical example

We now illustrate our results on a magnetic suspension system presented in [7]. The system is a laboratory device that consists of a coil whose voltage is controlled by a rather simple (non-reliable) pulse-width modulation system, and sensors for position of a suspended metallic sphere and for the coil current. The model around the origin without jumps and noise is in the form $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$, with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1750 & 0 & -34.1 \\ 0 & 0 & -0.0383 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1.9231 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The components of vector $x(t)$ are the position of the sphere, its speed and the coil current. The coil voltage $u(t)$ is controlled using a stabilizing state feedback control, leading to the closed loop dynamics $\dot{x}(t) = A_1x(t)$,

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1750 & 0 & -34.1 \\ 4360.2 & 104.2 & -84.3 \end{pmatrix}.$$

We consider the realistic scenario where the system may be operating in normal mode $\theta = 1$ or in critical failure $\theta = 2$ due e.g. to faults in the pulse-width modulation system, which is included in the model by making $B_2 = 0$, leading to the closed loop dynamics $\dot{x}(t) = A_2x(t)$ with $A_2 = A$. Although it is natural to consider that the system starts in normal mode a.s. and never recovers from a failure, we want to compare the performance of the proposed filter with the LMMSE [16] that requires a true Markov chain with positive probabilities for all modes at all times, then we relax the problem by setting the initial distribution $\pi(0) = (0.999, 0.001)$ and the transition rates matrix

$$\Lambda = \begin{pmatrix} -20 & 20 \\ 0.1 & -0.1 \end{pmatrix}$$

with the interpretation that the recovery from failure mode is relatively slow.

In the overall model Eq. (2) we set $C_1 = C_2 = C$ and we also consider that $x(0)$ is normally distributed with mean $\mathbb{E}[x(0)] = (0.001, 0, 0)'$ and variance $\text{Var}(x(0)) = I_3$,

$$E_1 = E_2 = \begin{pmatrix} 1 & 0.2 & -1.9 \\ -0.1 & 1.4 & -0.3 \\ 0.1 & 0.5 & 1 \end{pmatrix}, \quad D_1 = D_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so that only the position of the sphere and the coil current are measured through some noise. Speed is not observed. It is worth mentioning that the system is not mean square stable, so that the time horizon T is usually short for the trajectory to stay close to the origin and keep the linearized model valid; we can slightly increase the horizons during simulations for academic purposes only.

Number of grid points	Error for $\theta(0) = 1$	Error for $\theta(0) = 2$
10	5.441×10^{-3}	1017×10^{-3}
50	1.585×10^{-3}	357.5×10^{-3}
100	0.753×10^{-3}	175.2×10^{-3}
500	0.173×10^{-3}	36.22×10^{-3}
1000	0.100×10^{-3}	23.35×10^{-3}

Table 1: Quantization error for the first jump time depending on the number of points in the discretization grid and the value of the starting point of the Markov chain.

5.1 Markovian linear minimum mean squares estimator

Fragoso and Costa proposed in [16] the so-called Markovian linear minimum mean squares estimator (LMMSE) for MJLS with finite state space Markov chains. Under Assumption 2.1, the equation of the filter is

$$d\hat{x}_{FC}(t) = A_{\theta(t)}\hat{x}_{FC}(t)dt + K_{FC}(\theta(t), t)(dy(t) - C_{\theta(t)}\hat{x}_{FC}(t)dt),$$

for $0 \leq t \leq T$, with initial condition $\hat{x}_{FC}(0) = \mathbb{E}[x(0)]$ and gain matrices

$$K_{FC}(i, t) = P_{FC}(i, t)C'_i(D_i D'_i \pi_i(t))^{-1},$$

where $\pi_i(t) = \mathbb{P}(\theta(t) = i) = (\pi(0) \exp(t\Lambda))_i$ and $\{P_{FC}(i, t), 0 \leq t \leq T\}$ satisfies the system of matrix differential equation

$$\begin{aligned} dP_{FC}(i, t) &= (A_i P_{FC}(i, t) + P_{FC}(i, t) A'_i \\ &\quad + \sum_{j=1}^N P_{FC}(j, t) \Lambda_{ji} + E_i E'_i \pi_i(t) \\ &\quad - P_{FC}(i, t) C'_{\theta(t)} (D_{\theta(t)} D'_{\theta(t)} \pi_i(t))^{-1} \\ &\quad \times C_{\theta(t)} P_{FC}(i, t)) dt, \\ P_{FC}(i, 0) &= \text{Var}(x(0)) \pi_i(0). \end{aligned}$$

The matrices $\{P_{FC}(i, t), 0 \leq t \leq T, i \in \mathcal{S}\}$ and $\{K_{FC}(i, t), 0 \leq t \leq T, i \in \mathcal{S}\}$ depend only on the law of $\{\theta(t), 0 \leq t \leq T\}$ and not on its current value. Therefore they can be computed off line on a discrete time grid and stored but it is sub-optimal compared to the KBF.

5.2 Approximate filter by quantization

We start with the quantized discretization of the inter-jump times $\{S_n\}$ of the Markov chain $\{\theta(t), 0 \leq t \leq T\}$. We use the CLVQ algorithm described for instance in [27]. Table 1 gives the error $\mathbb{E}[|S_1 - \hat{S}_1|^2 \mid \theta(0) = i]^{1/2}$ for $i = 1, 2$ computed with 10^6 Monte Carlo simulations for an increasing number of discretization points. This illustrates the convergence of Theorem 3.1: the error decreases as the number of points increases. The variance of the first jump time in mode 2 is much higher than in mode 1 which accounts for the different scales in the errors.

The second step consists in solving the Riccati equation (2) for all possible trajectories of $\{\theta(t), 0 \leq t \leq T\}$ with inter-jump times in the quantization grids and up to the computation horizon $T = 0.02$. Namely, we compute the trajectories $\{\hat{P}_k(t), 0 \leq t \leq T\}$. We chose a

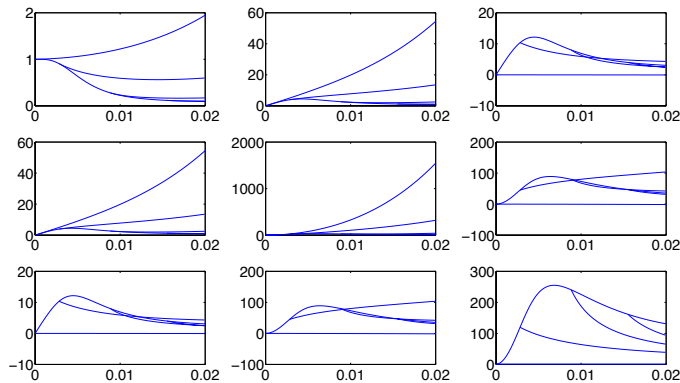


Figure 1: Pre-computed tree of solutions with 10 grid points.

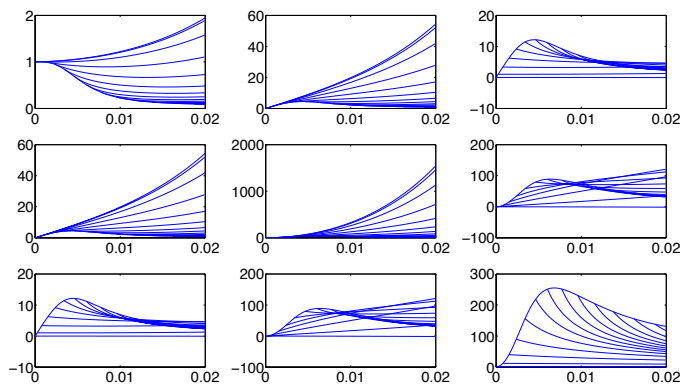


Figure 2: Pre-computed tree of solutions with 50 grid points.

regular time grid with time step $\delta t = 10^{-4}$. For technical reasons related to the selection of branches, the time horizon T is added in each grid. One thus obtains a tree of pre-computed branches that are solutions of Eq. (2), the branching times being the quantized jump times. Figures 1 and 2 show the pre-computed trees of solutions component-wise for 10 and 50 points respectively in the quantization grids. Note the very different scales of the coordinates. The number of grid points that are actually used (quantized points below the horizon T) are given in Table 2 for each original quantization grid size, together with the resulting number of pre-computed branches. The number of pre-computed branches grows exponentially fast when we take into account more grid points. Time taken to pre-compute the branches grows accordingly. In this example, the number of points used in mode 2 is low, therefore the number of branches remains tractable.

To compute the filtered trajectory in real time, one starts with the approximation of the solution of Eq. (2). The first branch corresponds to the pre-computed branch starting at time 0 from $\theta(0)$. When the first jump occurs, one selects the nearest neighbor of the jump time in the quantization grid and the corresponding pre-computed branch, and so on for the following jumps. Figure 3 shows the mean of the relative error between the solution of Eq (2) and its approximation (for the matrix norm 2) for given numbers of points in the quantization grids and 10^5 Monte Carlo simulations. Again, it illustrates how the accuracy of the approximation increases with the number of points in the quantization grids.

Finally, the real-time approximation of Eq (2) is plugged into the filtering equations to obtain an approximate KBF. Figure 4 shows the mean L^2 distance between the real KBF $\{\hat{x}_{KB}(t), 0 \leq t \leq T\}$ and its approximation $\{\tilde{x}, 0 \leq t \leq T\}$ following our procedure for an

Number of grid points	Points below horizon for $\theta(0) = 1$	Points below horizon for $\theta(0) = 2$	Number of branches
10	4	1	7
50	14	1	17
100	33	1	36
500	161	2	7763
1000	319	3	603784

Table 2: Number of grid points actually used and corresponding number of pre-computed branches depending on the initial number of points in the discretization grid.

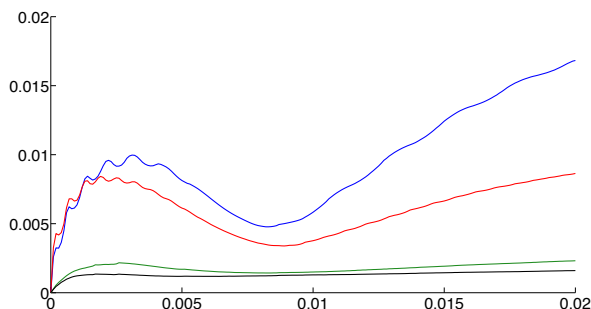


Figure 3: Average relative error between the solution of Riccati equation and its approximation, from top to bottom: blue: 50 points, red: 100 points, green: 500 point, black: 1000 points in the quantization grids.

increasing number of points in the quantization grids and for 10^5 Monte Carlo simulations.

5.3 Comparison of the filters

For each filter, we ran 10^5 Monte Carlo simulations and computed the mean of the following error between the real trajectory $\{x(t), 0 \leq t \leq T\}$ and the filtered trajectory $\{\hat{x}(t), 0 \leq t \leq T\}$ for all of the three filters presented above, the exact Kalman–Bucy filter being the reference.

$$\int_0^T \left((x_1(t) - \hat{x}_1(t))^2 + (x_2(t) - \hat{x}_2(t))^2 + (x_3(t) - \hat{x}_3(t))^2 \right) dt.$$

Table 3 gives this error for given numbers of points in the quantization grids. Of course only the error for the approximate filter changes with the quantization grids. Note that our approximate filter is very close to the KBF and performs better than the LMMSE for as little as 10 points in the quantization grids corresponding to 7 precomputed branches. We also ran our simulations with longer horizons. The performance of the filters is given in Table 4 and illustrate that our filter can still perform good with a longer horizon. Note that the computations of the LMMSE is impossible from an horizon of 0.4 on because the estimated state space reaches too high values very fast, and they are treated as infinity numerically. From an horizon of 0.8 on, all computations are impossible because the system is not mean square stable, as we explained before.

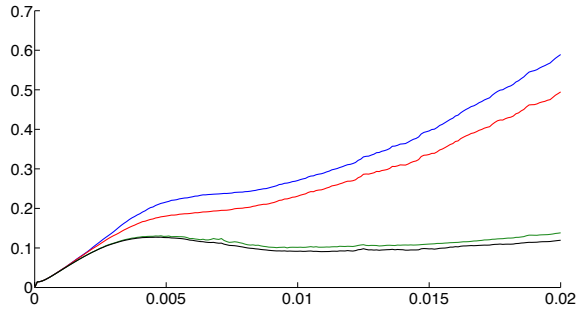


Figure 4: L^2 norm of the difference between \hat{x}_{KB} and its quantized approximation \tilde{x} , from top to bottom: blue: 50 points, red: 100 points, green: 500 point, black: 1000 points in the quantization grids.

Number of grid points	Error for KBF	Error for approximate filter	Error for LMMSE
10	3.9244	3.9634	3.9850
50	3.9244	3.9254	3.9850
100	3.9244	3.9246	3.9850
500	3.9244	3.9244	3.9850
1000	3.9244	3.9244	3.9850

Table 3: Average error for the different filters depending on the number of points in the quantization grids, considering horizon $T = 0.02$.

6 Conclusion

We have presented a filter for state estimation of sMJLS relying on discretization by quantization of the semi-Markov chain and solving a finite number of filtering Riccati equations. The difference between the approximated Riccati solution $\tilde{P}(t)$ and the actual Riccati solution $P(t)$ has been studied and we have shown in Theorem 4.6 that it converges to zero in average when the number of points in the discretization grid goes to infinity; a convergence rate is also provided, allowing to find a convergence rate for the gain matrices, see Corollary 4.8. Based on this result, and on an upper bound for the conditional second moment of the KBF that is derived in Lemma 4.11, we have obtained the main convergence result in Theorem 4.12, which implies convergence to zero of $\mathbb{E}|x_{KB}(t) - \tilde{x}(t)|^2$, so that $\tilde{x}(t)$ approaches $x_{KB}(t)$ almost surely as the number of grid points goes to infinity. Applications in which θ is not instantaneously observed can also benefit from the proposed filter, however it may not completely recover the performance of the KBF as explained in Remark 4.13. The algorithm has been applied to a real-world system and performed almost as well as the KBF with a small grid of 10 points.

Although the proposed filter can be pre-computed, the number of branches of the Riccati equation grows exponentially with the time horizon T , making the pre-computation time too high in some cases. One exception comprises systems with no more than one fast mode (high transition rates), because in such a situation the slow modes do not branch much and the number of branches grows in an almost linear fashion with T as long as the probability of the slow mode to jump before T remains small. Examples of applications coping with this setup, which can benefit from the proposed filter, are systems with small probability of failure and quick recovery (the failure mode is fast), or a variable number of permanent failures (the normal mode is fast), with web-based control as a fertile field of applications. For general systems, one possible way out of this cardinality issue is to use

T	Grid points	Branches	Error for KBF	Error for approx. filter	Error for LMMSE
0.1	10	12	376.3	425.6	812.5
0.1	50	110	376.3	379.1	812.5
0.1	100	3519	376.3	376.6	812.5
0.2	10	14	8597	10610	13260
0.2	50	2832	8597	9715	13260
0.3	10	14	2.325×10^4	4.893×10^6	3.023×10^5
0.3	50	11248	2.325×10^4	4.141×10^6	3.023×10^5
0.4	10	14	4.913×10^4	4.663×10^{10}	NaN
0.4	50	50049	4.913×10^4	2.102×10^{10}	NaN

Table 4: Average error for the different filters depending on the horizon, the number of points in the quantization grids and the number of branches.

a rolling-horizon scheme where the approximate gains are pre-computed in small batches during the system operation and sent to the controller memory. Another approach could be to quantize directly the sequence $\{S_k, P_k(S_k)\}$ thus keeping the number of branches at a fixed number, allowing for general transition rate matrices and longer horizons in terms of the number of jumps. However this approach suffers from a curse of dimensionality as the quantization error goes to zero with slower and slower rate as the dimension of the process goes higher, see Theorem 3.1.

Future work will look into a rolling-horizon implementation scheme, implementation issues and different compositions of the KBF/LMMSE, for instance using time-delayed solutions of the KBF that can be computed during the system operation as a measure for discarding unnecessary branches. Alternative schemes for discretization/quantization and selection of the appropriate pre-computed solutions can be pursued, seeking to reduce the computational load of the current algorithm while preserving the quality of the estimate.

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